ON CUBIC DIOPHANTINE EQUATION $x^2 + y^2 - xy = 39 z^3$

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Abstract

Four different methods of the non-zero non-negative solutions of non-homogeneous cubic Diophantine equation $x^2 + y^2 - xy = 39z^3$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

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1. INTRODUCTION

The number theory is the king of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For a broad review of variety of problems, one may try to see [3-12]. In this work, we are observed another interesting four different methods of the non-zero non-negative solutions the non-homogeneous cubic Diophantine equation $x^2 + y^2 - xy = 39z^3$ Further, some elegant properties among the special numbers and the solutions are observed.

2. DESCRIPTION OF METHOD

Consider the cubic Diophantine equation $x^2 + y^2 - xy = 39 z^3 \tag{1}$

Take the linear transformations $x = u + v$, $y = u - v$, $u \neq v \neq 0 \tag{2}$

Using (1) in (2), it gives $u^2 + 3v^2 = 39z^3 \tag{3}$

If we take $z = (a, b) = a^2 + 3b^2 = (a + i \sqrt{3} b) (a - i \sqrt{3} b), \tag{4}$

where $a$ and $b$ non-zero non-negative different integers, then we solve (1) through dissimilar method of solutions of (1) which are furnished below.

2.1 Method: I

We can write 39 as $39 = (6 + i \sqrt{3}) (6 - i \sqrt{3}) \tag{5}$

Using (4) and (5) in (3) and this gives

$$(u + i \sqrt{3} v) (u - i \sqrt{3} v) = (6 + i \sqrt{3}) (6 - i \sqrt{3})$$

$$(a + i \sqrt{3} b) (a - i \sqrt{3} b) \tag{6}$$

It gives us

$$(u + i \sqrt{3} v) = (6 + i \sqrt{3}) (a + i \sqrt{3} b)$$

$$(u - i \sqrt{3} v) = (6 - i \sqrt{3}) (a - i \sqrt{3} b) \tag{7}$$

Comparing both sides of (6) or (7), we obtain

$u = u(a, b) = 6a^3 - 36ab^2 - 6a^2b + 9b^3$$

$v = v(a, b) = a^3 - 6ab^2 + 12a^2b - 18b^3$\tag{8}$

In sight of (2), the solutions $x, y$ are found to be

$x = x(a, b) = 7a^3 - 42ab^2 + 6a^2b - 9b^3$\tag{9}$

$y = y(a, b) = 5a^3 - 30ab^2 - 18a^2b + 27b^3$\tag{9}$

Hence (4), (8) and (9) gives us two parametric the non-zero different integral values of (1).

Observations:

1. $y(b, b) = x (b, b) - 136b^5 + 68t_{b, b} = 0 \tag{1}$

2. $z(a, a) - 4t_{a, d} = 0 \tag{2}$

3. $x(a, 1) - 13P_{a} + t_{a, 3G_{21a}} + 8 = 0 \tag{3}$

4. $x(1,a) + 18P_{a} + 33t_{a, 3G_{21a}} = 0 \tag{4}$

5. $y(1,a) - 54P_{a} + 57t_{a, 3G_{21a}} = 0 \tag{5}$

6. $x(3b, 3b) + 10P_{a} + 5t_{b, b} = 0 \tag{6}$

7. $y(2a, 2a) + 1376 P_{a} + 688 t_{b, b} = 0 \tag{7}$

8. $z(a, 2a) + 13t_{a, b} = 0 \tag{8}$

2.2 method: II

We also write 39 as

$$39 = \frac{(9 + i5\sqrt{3})(9 - i5\sqrt{3})}{4} \tag{10}$$
Using (4) and (10) in (3) we obtain

\[(u+i\sqrt{3}v)(u-i\sqrt{3}v) = [(9+i5\sqrt{3})(9-i5\sqrt{3})]
\]
\[(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3]\]

Comparing both sides of above, we are found to be

\[(u+i\sqrt{3}v) = \frac{1}{2} [(9+i5\sqrt{3})(a+i\sqrt{3}b)^3]]
\]
\[(u-i\sqrt{3}v) = \frac{1}{2} [(9-i5\sqrt{3})(a-i\sqrt{3}b)^3]]
\]

Comparing both sides of (11) or (12), we obtain

\[u = u(a, b) = \frac{1}{2} [9a^3 - 54ab^2 - 30a^2b + 45b^3]
\]
\[v = v(a, b) = \frac{1}{2} [5a^3 - 30ab^2 + 18a^2b - 27b^3]\]

In true of (2), the values of x, y are given by

\[x = x(a, b) = 7a^3 - 42ab^2 - 6a^2b + 9b^3\]
\[y = y(a, b) = 2a^3 - 12ab^2 - 24a^2b + 36b^3\]

Hence (4), (13) and (14) gives us two parametric the non-zero different integral values of (1).

**Observations:**

1. \(y(a, b) - x(a, b) - 76a^2 + 38t_{4a, b} = 0\)
2. \(2x(b, b) - 9y(b, b) + 164P_b^5 - 82t_{4b, b} = 0\)
3. \(x(a, 1) - 14P_b^5 + 13t_{4a, a} + G_{12a} = 0 \pmod{29}\)
4. \(y(1, b) - 72P_b^5 + t_{32b} + 23t_{4b, b} = 0 \pmod{2}\)
5. \(z(2, b) - 4t_{4b, b} = 0\)
6. \(x(b, 1) = 10 P_b^5 + 23t_{3b} + G_{15b, b} = 0 \pmod{2}\)
7. \(z(6a, 6a) - 14t_{4a, a} = 0\)
8. \(x(2b, 3b) - 3 P_b^5 + 3 t_{4b, b} = 0\)

**2.3 Method: III**

Let us take (3) as \(u^2 + 3v^2 = 39z^3 + 1\)

Consider 1 as \(1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}\)

Using (4), (5) and (16) in (15) and it gives us

\[(u+i\sqrt{3}v)(u-i\sqrt{3}v) = \frac{1}{4} [(1+i\sqrt{3})(1-i\sqrt{3})]^3\]
\[(6+i\sqrt{3})(6-i\sqrt{3})(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3]\]

It gives us

\[(u+i\sqrt{3}v) = \frac{1}{2} [(1+i\sqrt{3})(6 + i\sqrt{3})(a + i\sqrt{3}b)]\]
\[(u-i\sqrt{3}v) = \frac{1}{2} [(1-i\sqrt{3})(6-i\sqrt{3})(a-i\sqrt{3}b)^3]\]

Comparing both sides of (17) or (18), we obtain

\[u = u(a, b) = \frac{1}{2} [3a^3 - 18ab^2 - 42a^2b + 63b^3]\]
\[v = v(a, b) = \frac{1}{2} [7a^3 - 42ab^2 + 6a^2b - 9b^3]\]

In sight of (2), the values of x, y are given by

\[x = x(a, b) = 5a^3 - 30ab^2 - 18a^2b + 27b^3\]
\[y = y(a, b) = -2ab^3 + 12ab^2 - 24a^2b + 36b^3\]

**Observations:**

1. \(x(1, b) + 57t_{4b} - 54P_b^5 - G_{12b} = 0 \pmod{2}\)
2. \(3x(b, b) + 5y(b, b) - 156P_b^5 + 78t_{4b, b} = 0\)
3. \(z(b, b) - 4t_{4b, b} = 0\)
4. \(y(1, b) - 72P_b^5 + G_{12b} + 48t_{4b, b} + 1 = 0\)
5. \(x(b, 1) = 10 P_b^5 + 23t_{3b} + G_{15b, b} = 0 \pmod{2}\)
6. \(z(6a, 6a) - 14t_{4a, a} = 0\)
7. \(x(2b, 3b) - 3 P_b^5 + 3 t_{4b, b} = 0\)

**2.4 Method: IV**

Replace (16) by \(1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}\)

Using (4), (10) and (21) in (15) and this gives us

\[(u+i\sqrt{3}v)(u-i\sqrt{3}v) = \frac{1}{4} \times 49 ((1+i4\sqrt{3})(1-i4\sqrt{3})(9-i5\sqrt{3})]\]
\[(9+i5\sqrt{3})(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3]\]

This gives us

\[(u+i\sqrt{3}v) = \frac{1}{14} [(1+i4\sqrt{3})(9+i5\sqrt{3})(a+i\sqrt{3}b)^3]\]
\[(u-i\sqrt{3}v) = \frac{1}{14} [(1-i4\sqrt{3})(9-i5\sqrt{3})(a-i\sqrt{3}b)^3]\]

Comparing both sides of (22) or (23), it gives us

\[u = u(a, b) = \frac{1}{14} [51a^3 + 306ab^2 - 246a^2b + 369b^3]\]
\[v = v(a, b) = \frac{1}{14} [41a^3 - 246ab^2 - 102a^2b + 153b^3]\]

In true of (2), the values of x, y are given by

\[x = x(a, b) = \frac{1}{7} [-5a^3 + 30ab^2 - 174a^2b + 261b^3]\]
\[y = y(a, b) = \frac{1}{7} [-46a^3 + 276ab^2 - 72a^2b + 108b^3]\]
Since our intention is to find integer solutions, taking $a$ as 7a and $b$ as 7b in (4), (24) and (25), the related parametric integer values of (1) are found as

$x = x(a, b) = -245a^2 + 1470ab^2 - 8526a^2b + 12789b^3$

$y = y(a, b) = 2254a^2 + 13524ab^2 - 3528a^2b^5 + 5292b^3$

$z = z(a, b) = 49a^2 + 147b^2$

**Observations:**

1. $x(a, a) - y(a, a) + 24108P_5 - 12054 t_{4,a} = 0$
2. $z(a, a) - t_{394,a} - G_{972} - P_a - 1 + t_{4,a} = 0$
3. $x(1, b) + 244 + G_{4263} - 5578P_5 + 1319 t_{4,b} = 0$
4. $y(a, 1) - 2508P_5 + 5782 t_{4,a} - G_{6762a} - 5293 = 0$
5. $z(a, 1) - 49t_{4,a} = 0 (\text{Mod 7})$
6. $x(b, 1) + 490 P_b - 8381 t_{4,b} - G_{775, b}$ = 0 (Mod 2)
7. $y(b, 1) - 4508 P_b + 5782 t_{4,b} - G_{6762, b} - 5293 = 0$
8. $z(3a, 3a) - 864 t_{4,a} = 0$

**3. CONCLUSION**

Here we observed various process of determining infinitely a lot of non-zero different integer values to the cubic Diophantine equation $x^2 + y^2 - xy = 39z^2$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

**4. REFERENCES**